**UNITED STATES AIR FORCE
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On a Volume Integral Equation Used in Solving 3-D Electromagnetic Interior Scattering Problems

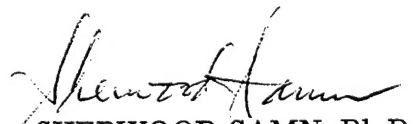Sherwood Samn

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For microwave dosimetry an accurate mathematical model of interior scattering in 3-D is needed. Human body is highly inhomogeneous. A formulation of the scattering problem based on a volume integral equation and which addressed inhomogeneity in detail is derived. As a by-product, this work provides credence to a certain heuristically derived formulation.

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ON A VOLUME INTEGRAL EQUATION USED IN SOLVING 3-D ELECTROMAGNETIC INTERIOR SCATTERING PROBLEMS

Introduction

Given an incident field $\mathbf{E}^{inc}$ generated from, for example, an antenna or a radar, the problem is to estimate the amount of electromagnetic energy deposition in a nearby human object. While this is a classical interior scattering problem, a versatile and efficient method for solving it in realistic settings is not generally available [1]. Some recent research in this area can be found in [3, 4, 5] and their references. Understanding the details of electromagnetic deposition in humans is essential for health and safety exposure considerations. Knowledge of deposition in humans and non-human animals is very important to medical research on the bioeffects of radiation exposure.

An approach to solving interior scattering problems is the integral equation approach. One of the advantages of this approach is that the boundary conditions of the problem are automatically incorporated into the integral equation, a feature that is especially useful in solving the corresponding inverse problem. For simple problems, the equivalence between the classical Maxwell's equations approach and the integral equation approach can easily be demonstrated. However, for general 3-D problem where the scatterers may have internal discontinuity in their material properties, this equivalence is no longer obvious. The purpose of this paper is to re-examine the validity and applicability of a formulation of a volume integral equation approach commonly used for solving interior scattering problems. It will be shown that a popular heuristic derivation of a fundamental volume integral equation is wanting, as it is mathematically incomplete. A more complete analysis of the volume integral equation formulation is given. As a by-product, the applicability of this approach is clarified. It is concluded that this approach can be applied to a broad class of interior scattering problems including the calculation of electromagnetic energy deposition in arbitrary three-dimensional inhomogeneous objects commonly encountered in the biomedical sciences.

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Preliminary

We assume a dielectric body of finite extent in free space is occupying a region V_b and is radiated by an incident wave. The incident wave is produced by a source enclosed in another finite region V_s which we assume is far away from the body. The free space outside V_b and V_s will be denoted by V_o.

In each of the three regions V_b, V_s, and V_o, the electric field, $\tilde{\mathbf{E}}$, the magnetic field, $\tilde{\mathbf{H}}$, the displacement, $\tilde{\mathbf{D}}$, and the magnetic induction $\tilde{\mathbf{B}}$ satisfy the Maxwell's equations. We assume the fields are time harmonic with a time-dependent factor $e^{j\beta\omega t}$, where β is fixed at either 1 or -1. The body, possibly nonhomogeneous, is assumed to be isotropic, linear, and non-magnetic. Thus, if $\tilde{\mathbf{E}}(\mathbf{x}, t) = \Re\{\mathbf{E}(\mathbf{x})e^{j\beta\omega t}\}$ and $\tilde{\mathbf{H}}(\mathbf{x}, t) = \Re\{\mathbf{H}(\mathbf{x})e^{j\beta\omega t}\}$ then $\mathbf{E}$ and $\mathbf{H}$ satisfy:

$$\nabla \times \mathbf{E} + j\beta\omega\mu_o\mathbf{H} = \mathbf{0}$$
$$\nabla \times \mathbf{H} - j\beta\omega\varepsilon\mathbf{E} = \mathbf{J}$$

where

$$\varepsilon = \{\varepsilon_s, \varepsilon_o, \varepsilon_b\}$$
$$\mathbf{J} = \{\mathbf{J}_s, \mathbf{0}, \mathbf{J}_b\}$$

in $V = \{V_s, V_o, V_b\}$, respectively. Here ε_o, μ_o are the free space permitivity and permeability respectively. $\mathbf{J}$ is the current density, which when restricted to $V_s(V_b)$ is $\mathbf{J}_s(\mathbf{J}_b)$. Similarly, the permitivity in $V_s(V_b)$ is $\varepsilon_s(\varepsilon_b)$. Since we assume the body is linear, $\mathbf{J}_b = \sigma\mathbf{E}$. Hence, with the usual modification of ε_b to include the conductivity, σ, we can assume, without loss of generality,

$$\mathbf{J}_b = 0$$

The incident wave, being the only field present when the body is absent, satisfies the time-harmonic Maxwell's equations:

$$\nabla \times \mathbf{E}^{inc} + j\beta\omega\mu_o\mathbf{H}^{inc} = \mathbf{0}$$
$$\nabla \times \mathbf{H}^{inc} - j\beta\omega\varepsilon^{inc}\mathbf{E}^{inc} = \mathbf{J}^{inc}$$

where

$$\varepsilon^{inc} = \{\varepsilon_s, \varepsilon_o, \varepsilon_o\}$$
$$\mathbf{J}^{inc} = \{\mathbf{J}_s, \mathbf{0}, \mathbf{0}\}$$

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in $V = \{V_s, V_o, V_b\}$, respectively. (We are making the tacit assumption that $\mathbf{J}_s$ is not changed by the presence of the scatterer (body): $\mathbf{J}^{inc} = \mathbf{J}$ in V_s.) Defining the scattered field $\mathbf{E}^s$ as $\mathbf{E} - \mathbf{E}^{inc}$ and similarly $\mathbf{H}^s$ as $\mathbf{H} - \mathbf{H}^{inc}$, then (recall $\mathbf{J}_b = \mathbf{0}$)

$$\nabla \times \mathbf{E}^s + j\beta\omega\mu_o\mathbf{H}^s = \mathbf{0} \tag{1}$$
$$\nabla \times \mathbf{H}^s - j\beta\omega\left(\varepsilon\mathbf{E} - \varepsilon^{inc}\mathbf{E}^{inc}\right) = \mathbf{0} \tag{2}$$

Except possibly in V_s (which we will not consider), this system can be written as

$$\nabla \times \mathbf{E}^s + j\beta\omega\mu_o\mathbf{H}^s = \mathbf{0} \tag{3}$$
$$\nabla \times \mathbf{H}^s - j\beta\omega\varepsilon_o\mathbf{E}^s = \mathbf{J}_{eq} \tag{4}$$

where

$$\mathbf{J}_{eq} = j\beta\omega\left(\varepsilon - \varepsilon^{inc}\right)\mathbf{E}$$

From the definitions of ε and ε^{inc}, it is clear that $\mathbf{J}_{eq}$ vanishes identically everywhere outside V_b, including V_s.

Combining Equations (3) and (4), we get

$$\nabla \times \nabla \times \mathbf{E}^s - k_o^2\mathbf{E}^s = -j\beta\omega\mu_o\mathbf{J}_{eq} \tag{5}$$

where $k_o^2 = \varepsilon_o\mu_o\omega^2$.

Furthermore, using the vector identity $\nabla \times \nabla \times \mathbf{E}^s = \nabla(\nabla \cdot \mathbf{E}^s) - \nabla^2\mathbf{E}^s$, we also get the equation

$$\nabla^2\mathbf{E}^s + k_o^2\mathbf{E}^s = \mathbf{R} \tag{6}$$

where $k^2(\mathbf{r}) = \varepsilon(\mathbf{r})\mu_o\omega^2$ and

$$\mathbf{R}(\mathbf{r}) = -\left(k^2(\mathbf{r}) - k_0^2\right)\mathbf{E}(\mathbf{r}) + \nabla(\nabla \cdot \mathbf{E}(\mathbf{r}))$$

To solve any of these equivalent equations for the scattered field uniquely, we need to specify the boundary conditions it must satisfy. Besides requiring $\mathbf{E}^s$ to satisfy the radiation condition, we also require the tangential components of $\mathbf{E}^s$ and $\mathbf{H}^s$ (or equivalently, $\nabla \times \mathbf{E}^s$, since we are assuming a non-magnetic body) to be continuous across S_b, the boundary of V_b:

$$\left(\hat{\mathbf{n}}(\mathbf{r}) \times \mathbf{E}^s(\mathbf{r})\right)_+ = \left(\hat{\mathbf{n}}(\mathbf{r}) \times \mathbf{E}^s(\mathbf{r})\right)_- \tag{7}$$
$$\hat{\mathbf{n}}(\mathbf{r}) \times \left(\nabla \times \mathbf{E}^s(\mathbf{r})\right)_+ = \hat{\mathbf{n}}(\mathbf{r}) \times \left(\nabla \times \mathbf{E}^s(\mathbf{r})\right)_- \tag{8}$$

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This follows from the boundary conditions on the **E** and **H** and the fact that $\mathbf{E}^{inc}$ is continuous across S_b. In the last set of equations, quantities with a positive (negative) subscript denote values on the exterior (interior) boundary.

A Heuristic Formulation

In the unbounded free space, it is well-known that the vector equation

$$\nabla \times \nabla \times \mathbf{F} - k_o^2 \mathbf{F} = -j\beta\omega\mu_o \mathbf{J} \tag{9}$$

has a unique solution satisfying the radiation condition. In fact, it is given by

$$\mathbf{F}(\mathbf{r}) = \beta_o \left(\bar{\mathbf{I}} + \frac{1}{k_o^2}\nabla\nabla\right) \cdot \int_V g(\mathbf{r},\mathbf{r}')\, \mathbf{J}(\mathbf{r}')\, dr' \tag{10}$$

where V is the support of **J** and $\beta_0 = -j\beta\omega\mu_0$. Here the function g is the free-space Green function for the three-dimensional scalar wave equation. That is to say it satisfies the equation

$$\nabla^2 g(\mathbf{r},\mathbf{r}') + k_o^2 g(\mathbf{r},\mathbf{r}') = -\delta(\mathbf{r}-\mathbf{r}') \tag{11}$$

The function g is explicitly given by

$$g(\mathbf{r},\mathbf{r}') = \frac{e^{-j\beta k_o|\mathbf{r}-\mathbf{r}'|}}{4\pi \mid \mathbf{r} - \mathbf{r}' \mid}, \tag{12}$$

Since Equation (5) has exactly the same form as Equation (9), it is often concluded therefore that the unique scattered field must also be given by:

$$\mathbf{E}^s(\mathbf{r}) = \beta_o \left(\bar{\mathbf{I}} + \frac{1}{k_o^2}\nabla\nabla\right) \cdot \int_V g(\mathbf{r},\mathbf{r}')\, \mathbf{J}_{eq}(\mathbf{r}')\, dr' \tag{13}$$

leading to an integral equation for the total field **E**:

$$\mathbf{E}(\mathbf{r}) - \left(\bar{\mathbf{I}} + \frac{1}{k_o^2}\nabla\nabla\right) \cdot \int_V g(\mathbf{r},\mathbf{r}')\, (k^2(\mathbf{r}') - k_o^2)\, \mathbf{E}(\mathbf{r}')\, dr' = \mathbf{E}^{inc}(\mathbf{r}) \tag{14}$$

While this reasoning is plausible, it is *mathematically incomplete*, as the issue of boundary conditions on S_b has not been addressed. In the following, we

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will give a full derivation of an integral equation for the electric field in the presence of a finite number of bounded scatterers in free space. We assume each scatterer is piecewise continuously differentiable in $\varepsilon(\mathbf{r})$. In this derivation the role of the boundary conditions, both physical and mathematical (due to discontinuity in ε within each scatterer), will be evident.

A Detailed Formulation

We assume the entire 3-dimensional space can be represented as the disjoint union of a finite number $(N+2)$ of sub-regions, V_k $(k = 0, \ldots, N+1)$. All sub-regions are bounded except one. V_o will be reserved for this unbounded sub-region and it represents the free space exterior to all the scatterers together with one (assumed for simplicity) sub-region V_{N+1} that contains the source that generates the incident field. Thus,

$$\mathbf{R}^3 = \biguplus_{k=0}^{k=N+1} V_k \tag{15}$$

We assume $\varepsilon(\mathbf{r})$ is continuously differentiable in each sub-region with possible discontinuity at the boundary. In general there will be more sub-regions than there are scatterers, as each scatterer may have to be represented by more than one sub-region on which $\varepsilon(\mathbf{r})$ is continuously differentiable. This formulation is quite general and includes the important piecewise homogeneous case, which is often used to model human organs or even the complete human body.

We start with a well-known variant of the Green's theorem involving a vector field $\mathbf{F}(\mathbf{r})$ and a scalar field $f(\mathbf{r})$. Assuming both fields are defined on a connected region V with a smooth boundary surface S and are continuously differentiable, then this particular variant of the Green's theorem takes the form:

$$\int_V \left[f\nabla^2 \mathbf{F} - \mathbf{F}\nabla^2 f \right] dV = \int_S [\, (\nabla \cdot \mathbf{F})f\hat{\mathbf{n}} - (\hat{\mathbf{n}} \cdot \mathbf{F})\nabla f$$
$$-\hat{\mathbf{n}} \times (\nabla \times \mathbf{F})f \quad - \quad (\hat{\mathbf{n}} \times \mathbf{F}) \times (\nabla f)\,] \, dS \tag{16}$$

Here $\hat{\mathbf{n}}$ is the unit vector normal to the surface S of V pointing into the exterior of V. When more specificity is required, this unit normal vector is also denoted by $\hat{\mathbf{n}}_s^V$.

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In each sub-region $V_k, (k = 0, \ldots, N)$, the electric field $\mathbf{E}$ satisfies Equation (6) also. In the source region V_{N+1}, $\mathbf{E}$ satisfies Equation (6) with a modified $\mathbf{R}$ ($\mathbf{R} + j\beta\omega\mu_o\mathbf{J}_s$) that accounts for the source that generates the incident field. In any case, if we let $V = V_k$, $\mathbf{F(r)} = \mathbf{E(r)}$, and $f(\mathbf{r}) = g(\mathbf{r,r'})$ in Equation (16), where $\mathbf{r'}$ is any point not on the boundary ∂V_k of V_k, we readily obtain, using Equations (6) and (11), the integral equation

$$\mathbf{E(r')}\,\delta(\mathbf{r'}, V_k) = -\int_{V_k}\mathbf{R(r)}\,g(\mathbf{r,r'})\,dV + \mathbf{S}_k(\mathbf{r'}) \tag{17}$$

where

$$\delta(\mathbf{r}, V) = \begin{cases} 1 & \text{for } \mathbf{r} \text{ in the interior of } V \\ 0 & \text{for } \mathbf{r} \text{ in the exterior of } V \end{cases} \tag{18}$$

and

$$\mathbf{S}_k(\mathbf{r'}) = \mathbf{Sd}(\mathbf{r'}; \partial V_k, V_k) + \mathbf{SG}(\mathbf{r'}, -1; \partial V_k, V_k) + \mathbf{X}_k(\mathbf{r'}) \tag{19}$$

and where

$$\mathbf{Sd}(\mathbf{r'}; \partial V_k, V_k) = \int_{\partial V_k} (\nabla \cdot \mathbf{E}_{V_k}(\mathbf{r}))\,g(\mathbf{r,r'})\hat{\mathbf{n}}^{V_k}_{\partial V_k}(\mathbf{r})\,dS$$

$$\mathbf{SG}(\mathbf{r'}, \alpha(*); \partial V_k, V_k) = \int_{\partial V_k} \alpha(\mathbf{r})\,(\hat{\mathbf{n}}^{V_k}_{\partial V_k}(\mathbf{r}) \cdot \mathbf{E}_{V_k}(\mathbf{r}))\,\nabla g(\mathbf{r,r'})\,dS$$

$$\mathbf{X}_k(\mathbf{r'}) = -\int_{\partial V_k} \hat{\mathbf{n}}^{V_k}_{\partial V_k}(\mathbf{r}) \times (\nabla \times \mathbf{E}_{V_k}(\mathbf{r}))g(\mathbf{r,r'})\,dS$$

$$- \int_{\partial V_k} (\hat{\mathbf{n}}^{V_k}_{\partial V_k}(\mathbf{r}) \times \mathbf{E}_{V_k}(\mathbf{r})) \times \nabla g(\mathbf{r,r'})\,dS \tag{20}$$

For $\mathbf{r}$ on the boundary ∂V_k, $\mathbf{E}_{V_k}(\mathbf{r}) = \lim_{\mathbf{r'}\to\mathbf{r}} \mathbf{E(r')}$ for $\mathbf{r'}$ in V_k. The same holds for $\alpha(\mathbf{r})$. By definition, $\mathbf{SG}(\mathbf{r'}, \alpha; \partial V_k, V_k)$ is linear in α. Hence,

$$\mathbf{SG}(\mathbf{r'}, -1; \partial V_k, V_k) = \mathbf{SG}(\mathbf{r'}, \frac{\varepsilon_{V_k}(*)}{\varepsilon_{V_o}} - 1; \partial V_k, V_k) + \mathbf{Y}_k(\mathbf{r'}) \tag{21}$$

where

$$\mathbf{Y}_k(\mathbf{r'}) = -\mathbf{SG}(\mathbf{r'}, \frac{\varepsilon_{V_k}(*)}{\varepsilon_{V_o}}; \partial V_k, V_k) \tag{22}$$

Equation (17) now becomes

$$\mathbf{E(r')}\,\delta(\mathbf{r'}, V_k) = -\int_{V_k}\mathbf{R(r)}\,g(\mathbf{r,r'})\,dV$$

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$$+ \quad \mathbf{Sd}(\mathbf{r}'; \partial V_k, V_k) + \mathbf{SG}\left(\mathbf{r}', \frac{\varepsilon_{V_k}(*)}{\varepsilon_{V_o}} - 1; \partial V_k, V_k\right)$$

$$+ \quad \mathbf{X}_k(\mathbf{r}') + \mathbf{Y}_k(\mathbf{r}') \tag{23}$$

The following proposition allows one to convert the two surface integrals $\mathbf{Sd}$ and $\mathbf{SG}$ in Equation (23) into volume integrals $\mathbf{GdV}$ and $\mathbf{VGd}$ defined as

$$\mathbf{GdV}(\mathbf{r}', \alpha(*); V) = \nabla'\nabla' \cdot \int_V \alpha(\mathbf{r})\mathbf{E}(\mathbf{r})\, g(\mathbf{r}, \mathbf{r}')\, dV$$

$$\mathbf{VGd}(\mathbf{r}'; V) = \int_V \nabla(\nabla \cdot \mathbf{E}(\mathbf{r}))\, g(\mathbf{r}, \mathbf{r}')\, dV \tag{24}$$

Proposition A. Suppose $\alpha(\mathbf{r})$ is continuously differentiable in a bounded region V with smooth boundary ∂V such that $\nabla \cdot (\alpha(\mathbf{r}) + 1)\, \mathbf{E}(\mathbf{r}) = 0$ in V, then

$$\mathbf{Sd}(\mathbf{r}'; \partial V, V) + \mathbf{SG}(\mathbf{r}', \alpha(*); \partial V, V) = \mathbf{GdV}(\mathbf{r}', \alpha(*); V) + \mathbf{VGd}(\mathbf{r}'; V) \tag{25}$$

Proof. This proposition basically follows from the application of the Divergence Theorem and the use of appropriate vector identities involving gradient and divergence. In particular, one can readily show

$$\int_{\partial V}(\nabla \cdot \mathbf{E})\,\hat{\mathbf{n}}\, g\, dS = \int_V \nabla(\nabla \cdot \mathbf{E})\, g + \int_V (\nabla g)\nabla \cdot \mathbf{E}\, dV$$

$$\int_{\partial V} \alpha\,(\hat{\mathbf{n}} \cdot \mathbf{E})\, \nabla g\, dS = \nabla'\nabla' \cdot \int_V \alpha \mathbf{E}\, g\, dV - \int_V (\nabla g)\nabla \cdot \mathbf{E}\, dV$$

$$- \quad \nabla' \int_V \mathbf{E} \cdot \nabla \alpha\, g\, dV$$

$$- \quad \nabla' \int_V (\alpha + 1)g\nabla \cdot \mathbf{E}\, dV \tag{26}$$

The proposition is proved by showing the sum of the last two integrals is zero because of the assumption on α.

Now summing Equation (23) over all sub-regions $V_k, (k = 0, \ldots, N + 1)$ and using Proposition A and the definition of $\mathbf{R}$ (appropriately modified for the

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source region), we readily obtain for $\mathbf{r}'$ not on $\partial V_k, (k = 0, \ldots, N + 1)$

$$
\begin{aligned}
\mathbf{E}(\mathbf{r}') \;=\; & \int_{V_b} (k^2(\mathbf{r}) - k_0^2) \, \mathbf{E}(\mathbf{r}) \, g(\mathbf{r}, \mathbf{r}') \, dV \\
& + \; \nabla' \nabla' \cdot \int_{V_b} \frac{(k^2(\mathbf{r}) - k_0^2)}{k_0^2} \, \mathbf{E}(\mathbf{r}) \, g(\mathbf{r}, \mathbf{r}') \, dV \\
& + \; \sum_{k=0}^{k=N+1} (\mathbf{X}_k(\mathbf{r}') + \mathbf{Y}_k(\mathbf{r}')) \\
& + \; \mathbf{E}^{rem}(\mathbf{r}') && (27)
\end{aligned}
$$

where V_b is the union of all the sub-regions representing the scatterers and $\mathbf{E}^{rem}$ is

$$
\begin{aligned}
\mathbf{E}^{rem}(\mathbf{r}') \;=\; & - \int_{V_{N+1}} (\mathbf{R}(\mathbf{r}) + j\beta\omega\mu_o\mathbf{J}_s(\mathbf{r})) \, g(\mathbf{r}, \mathbf{r}') \, dV \\
& + \; \mathbf{S}_{N+1}(\mathbf{r}') - \mathbf{X}_{N+1}(\mathbf{r}') - \mathbf{Y}_{N+1}(\mathbf{r}')
\end{aligned}
$$

Equation (27) is a representation of *any* solution of the Maxwell's Equations. To determine the unique solution to the scattering problem, we need to impose appropriate boundary conditions: Equations (7) and (8) and the radiation condition.

Proposition B. If $\mathbf{E}$ satisfies the boundary conditions in Equations (7) and (8) and the radiation condition, then

$$
\sum_{k=0}^{k=N+1} \mathbf{X}_k(\mathbf{r}') = 0 \qquad\qquad (28)
$$

Proof. The boundary ∂V_k of each sub-region $V_k, (k = 0, \ldots, N + 1)$ is made up of $n_k \geq 1$ smooth sub-surfaces $S_{k,i} \, (i = 1, \ldots, n_k)$. Since each sub-surface $S_{k,i}$, if it is bounded, is the surface of or is part of the surface of exactly two sub-regions, it appears exactly twice in the sum in (28). Suppose a sub-surface S is a surface between two sub-regions V_1 and V_2, then its total contribution C_X to the sum in (28) is

$$
C_X = - \int_S [\hat{\mathbf{n}}_s^{V_1}(\mathbf{r}) \times (\nabla \times \mathbf{E}_{V_1}(\mathbf{r})) \;+\; \hat{\mathbf{n}}_s^{V_2}(\mathbf{r}) \times (\nabla \times \mathbf{E}_{V_2}(\mathbf{r}))] \, g(\mathbf{r}, \mathbf{r}') \, dS
$$

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$$-\int_{S}[(\hat{\mathbf{n}}_{s}^{V_1}(\mathbf{r}) \times \mathbf{E}_{V_1}(\mathbf{r})) \;+\; (\hat{\mathbf{n}}_{s}^{V_2}(\mathbf{r}) \times \mathbf{E}_{V_2}(\mathbf{r}))] \times \nabla g(\mathbf{r},\mathbf{r}') \, dS$$

However, since $\hat{\mathbf{n}}_{s}^{V_1}(\mathbf{r}) = -\hat{\mathbf{n}}_{s}^{V_2}(\mathbf{r})$, it follows from the boundary conditions (7) and (8) that C_X vanishes. For the sole unbounded surface (the bounding surface of V_o at infinity), its contribution to the sum in (28) also vanishes due to the radiation condition.

Proposition C. If $\mathbf{E}$ satisfies the boundary condition in Equation (8) then

$$\sum_{k=0}^{k=N+1} \mathbf{Y}_k(\mathbf{r}') = 0 \qquad (29)$$

Proof. As in Proposition B, suppose a sub-surface S is a surface between two sub-regions V_1 and V_2, then its total contribution C_Y to the sum in (29) is

$$
\begin{aligned}
C_Y \;=\; & -\int_{S}[\frac{\varepsilon_{V_1}(\mathbf{r})}{\varepsilon_{V_o}}(\hat{\mathbf{n}}_{s}^{V_1}(\mathbf{r}) \cdot \mathbf{E}_{V_1}(\mathbf{r})) \\
& + \frac{\varepsilon_{V_2}(\mathbf{r})}{\varepsilon_{V_o}}(\hat{\mathbf{n}}_{s}^{V_2}(\mathbf{r}) \cdot \mathbf{E}_{V_2}(\mathbf{r}))] \, \nabla g(\mathbf{r},\mathbf{r}') \, dS
\end{aligned}
$$

As a *consequence* of Equation (8) [2], we have

$$\varepsilon_{V_1}(\mathbf{r})\,\hat{\mathbf{n}}_{s}^{V_1} \cdot \mathbf{E} = \varepsilon_{V_2}(\mathbf{r})\,\hat{\mathbf{n}}_{s}^{V_2} \cdot \mathbf{E}, \qquad (30)$$

the familiar boundary condition on the normal component of $\mathbf{D}$. (We could have made use of Equation (30) directly without referencing its connection with Equation (8), but this would have obscured the fact that the boundary conditions on the tangential components of $\mathbf{E}^s$ and $\mathbf{H}^s$ alone uniquely determined the field.)   Again, because $\hat{\mathbf{n}}_{s}^{V_1}(\mathbf{r}) = -\hat{\mathbf{n}}_{s}^{V_2}(\mathbf{r})$, C_Y vanishes.

Because of Propositions B and C, Equation (27) becomes

$$\mathbf{E}(\mathbf{r}') \;=\; \int_{V_b}(k^2(\mathbf{r}) - k_0^2)\,\mathbf{E}(\mathbf{r})\,g(\mathbf{r},\mathbf{r}')\,dV$$

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$$+ \quad \nabla'\nabla' \cdot \int_{V_b} \frac{(k^2(\mathbf{r}) - k_0^2)}{k_0^2} \, \mathbf{E}(\mathbf{r}) \, g(\mathbf{r}, \mathbf{r}') \, dV$$

$$+ \quad \mathbf{E}^{rem}(\mathbf{r}') \tag{31}$$

Finally, when $k^2(\mathbf{r}) = k_o^2(\mathbf{r})$, Equation (31) reduces to $\mathbf{E} = \mathbf{E}^{rem}$. Thus, $\mathbf{E}^{rem} = \mathbf{E}^{inc}$, by definition. Hence, for $\mathbf{r}'$ not on any boundary,

$$
\begin{aligned}
\mathbf{E}(\mathbf{r}') \quad = \quad & \int_{V_b} (k^2(\mathbf{r}) - k_0^2) \, \mathbf{E}(\mathbf{r}) \, g(\mathbf{r}, \mathbf{r}') \, dV \\
+ \quad & \nabla'\nabla' \cdot \int_{V_b} \frac{(k^2(\mathbf{r}) - k_0^2)}{k_0^2} \, \mathbf{E}(\mathbf{r}) \, g(\mathbf{r}, \mathbf{r}') \, dV \\
+ \quad & \mathbf{E}^{inc}(\mathbf{r}')
\end{aligned}
\tag{32}
$$

Which is exactly Equation (14).

Conclusion

We have given in this report a detailed derivation of a volume integral equation that can be used to solve interior scattering problems when the material property of the scatterer is piecewise continuously differentiable.

While it is natural to make the distinction between the physical (external) boundary of the scatterer and its mathematical (internal) boundaries (which arise when there is a discontinuity of material property), all physical boundaries are generally mathematical boundaries and vice versa. Thus, mathematical boundaries should be treated with as much care as with physical boundaries. Furthermore, different types of discontinuities may require different treatment. In fact, in Müller's classical text [2], he made a careful distinction between the case where the material property has a discontinuity in the derivative of $\epsilon(\mathbf{r})$ at the boundary and the case where it has a simple jump in $\epsilon(\mathbf{r})$ at the boundary. This suggests that care should be exercised when dealing with scatterers with piecewise continuously differentiable material property.

Indeed the volume integral equation we derived here is based on a careful treatment of boundary conditions. The derivation clearly shows the roles played by boundary conditions on both the physical boundary and mathematical boundary.

That our formulation turns out to be exactly the same as that derived heuristically (without addressing boundary conditions) is also reassuring. Our formulation is valid provided the observation point is not on a boundary, internal or otherwise.

Finally, with regard to implementation, Equation (32) is generally not used directly because of the difficulty with differentiating inexact data. However, because of the additivity of integration over volume, a Lippman-Schwinger type equation can be obtained by moving $\bar{I} + \nabla\nabla$ under the integral in Equation (32). This will be valid provided the observation point $\mathbf{r}$ is not on a boundary and that proper account is taken of the singularity of the resulting Dyadic Green's Function (in 3-D). Furthermore, numerical implementation of this formulation will likely be more accurate if fewer cells in the mesh straddle surfaces of discontinuity. Hence a mesh of tetrahedra would be more suitable than a mesh of rectangular blocks if the scatterer is highly inhomogeneous, as in the case of the human body.

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